Quantum Dynamical Entropy of Unitary Operators

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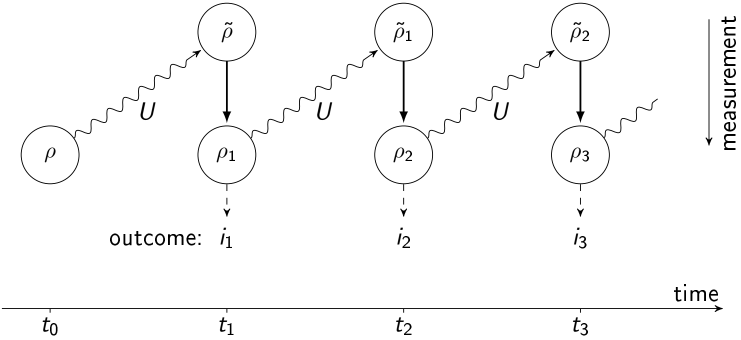
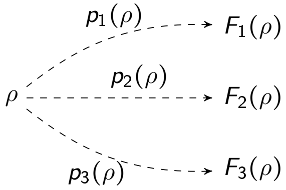
1. A d-dimensional quantum system subject to **successive measurements**.
2. Between each two measurements the system undergoes deterministic time evolution described by a **unitary** operator U.
3. There are k possible measurement outcomes.
4. We want to quantify the **irreducible randomness** of the sequences of outcomes.



For an input state ρ, a unitary U, and a POVM $\Pi = \{\Pi_1, \ldots, \Pi_k\}$:

- the probability of obtaining the result i: $\mathrm{p}_i(\rho)$
- the post-measurement state (if the outcome i has been obtained): $\mathrm{F}_i(\rho)$

Partial Iterated Function System (PIFS) generated by U and Π.



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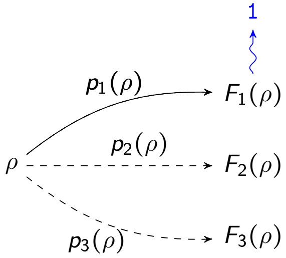

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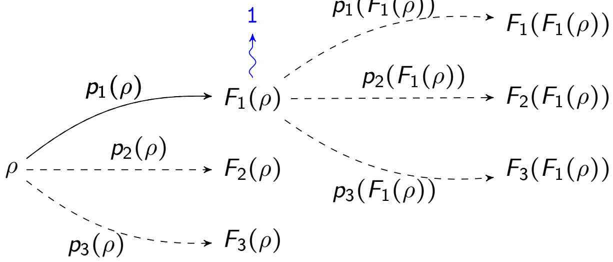

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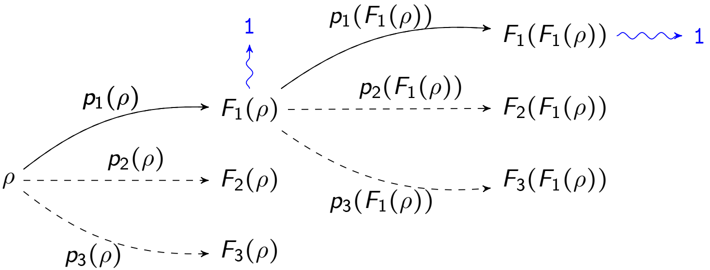

For an input state ρ, a unitary U, and a POVM $\Pi = \{\Pi_1, \ldots, \Pi_k\}$:

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- the post-measurement state (if the outcome i has been obtained): $\mathrm{F}_i(\rho) = \frac{\sqrt{\Pi_i}\, U \rho U^* \sqrt{\Pi_i}}{\mathrm{tr}(\Pi_i U \rho U^*)}$ (generalized Lüders instrument)

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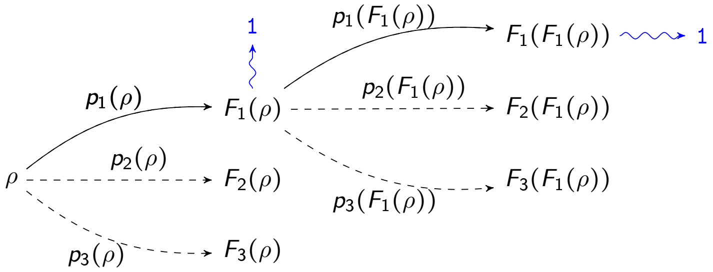

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Partial Iterated Function System (PIFS) generated by U and Π.

$\Rightarrow$ We get a Markov chain on quantum states.

$\Rightarrow$ Process induced by $\mathcal{F}_{U,\Pi}$ on $\{1, \ldots, k\}$ need not be Markovian.

$\Rightarrow$ Probability of outputting the string of outcomes $(i_1, \ldots, i_n)$:

$$\mathrm{p}_{i_1,\ldots,i_n}(\rho) = \mathrm{p}_{i_1}(\rho)\mathrm{p}_{i_2}(\mathrm{F}_{i_1}(\rho)) \cdot \ldots \cdot \mathrm{p}_{i_n}(\mathrm{F}_{i_{n-1}} \ldots \mathrm{F}_{i_1}(\rho)).$$

$\Rightarrow$ Evolution of Dirac delta measures on quantum states:

$$\Psi\colon\ \delta_\rho \longmapsto \sum_{\substack{i=1,\ldots,k \\ \mathrm{p}_i(\rho)>0}} \mathrm{p}_i(\rho)\, \delta_{\mathrm{F}_i(\rho)}.$$

***n*-th partial entropy**:

$$H_n := \sum_{i_1,\ldots,i_n=1}^{k} \eta\left(\mathrm{p}_{i_1,\ldots,i_n}(\rho_*)\right) \quad \text{where} \quad \eta(x) := \begin{cases} -x\ln x & x > 0 \\ 0 & x = 0 \end{cases}$$

Quantum dynamical entropy of *U* with respect to Π:

$$H(U,\Pi) := \lim_{n\to\infty} \frac{H_n}{n} = \lim_{n\to\infty} (H_{n+1} - H_n)$$

Blackwell integral formula (1957):

$$H(U,\Pi) = \int_{\mathcal{S}(\mathbb{C}^d)} H_1 \, \mathrm{d}\mu_*$$

where μ_* is the *- limit of $(\Psi^n(\delta_{\rho_*}) : n \in \mathbb{N})$.

Shannon (1948), Kolmogorov (1958);
Srinivas (1978), Pechukas (1982), Beck & Graudenz (1992) - for projective measurements; Słomczyński & Życzkowski (1994) - for generalized measurements; Crutchfield & Wiesner (2008) - *entropy rate*

rank-1 POVMs

Let Π consist of k one-dim (rescaled) projections: $\Pi_i = \frac{d}{k} |\varphi_i\rangle \langle\varphi_i|$

- Probabilities in the first step: $p_i(\rho_*) = \frac{1}{k}$
- Post-measurement state: $\mathsf{F}_i(\rho) = |\varphi_i\rangle \langle\varphi_i|$ for every ρ and i
- Probabilities in the subsequent steps: $p_j(\mathsf{F}_i(\rho)) = \frac{d}{k} \left|\langle\varphi_j|U|\varphi_i\rangle\right|^2$

$$\Rightarrow \quad \mu_* \text{ is uniform}, \qquad H(U, \Pi) = \frac{1}{k} \sum_{i,j=1}^{k} \eta\left(\frac{d}{k} \left|\langle\varphi_j|U|\varphi_i\rangle\right|^2\right)$$

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- $\mathsf{F}_1,\ldots,\mathsf{F}_k$ are constant: *one symbol* ~ *one quantum state*.
- The process on $\{1,\ldots,k\}$ is a Markov chain.

rank-1 PVMs

Π = projections on an orthonormal basis of $\mathbb{C}^d$

$$H(U,\Pi) = \frac{1}{d}\sum_{i,j=1}^{d} \eta\left(\left|\langle\varphi_i|U|\varphi_j\rangle\right|^2\right)$$

Quantum dynamical entropy of *U* (independent of measurement):

$$H^{dyn}(U) := \max\{H(U,\Pi) \colon \Pi \text{ is a rank-1 PVM}\}$$

$$= \max_{\substack{(\varphi_j)_{j=1}^d \\ \text{orthonormal} \\ \text{bases of } \mathbb{C}^d}} \frac{1}{d}\sum_{i,j=1}^{d} \eta\left(\left|\langle\varphi_i|U|\varphi_j\rangle\right|^2\right)$$

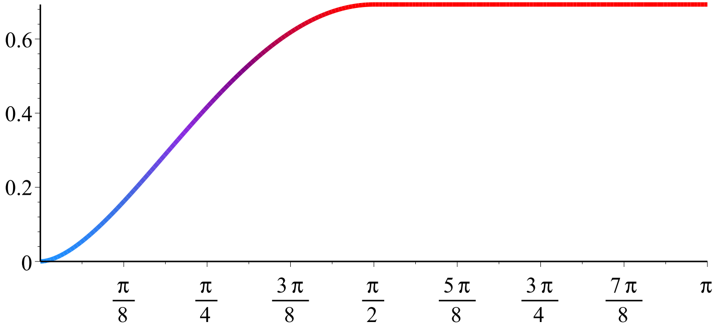
- $H^{\mathrm{dyn}}(U)$ depends only on the eigenvalues of U;
- $H^{\mathrm{dyn}}(U) = H^{\mathrm{dyn}}(e^{\mathrm{i}\varphi}U)$ for $\varphi \in \mathbb{R}$;
- $0 \le H^{dyn}(U) \le \ln d$.

Qubits

$$U_\theta \sim \begin{bmatrix} e^{\frac{\theta}{2}i} & 0 \\ 0 & e^{-\frac{\theta}{2}i} \end{bmatrix}$$

$\theta \in [0, \pi]$ is the smaller angle between the eigenvalues of U_θ

$$H^{\text{dyn}}(U_\theta) = \begin{cases} \eta\left(\frac{1+\cos\theta}{2}\right) + \eta\left(\frac{1-\cos\theta}{2}\right) & \theta \le \frac{\pi}{2} \\ \ln 2 & \theta \ge \frac{\pi}{2} \end{cases}$$



We say that a unitary $U \in \mathcal{U}(\mathbb{C}^d)$ is **chaotic** iff $H^{\mathrm{dyn}}(U) = \ln d$.

The following conditions are equivalent:

1. U is chaotic;
2. there exists an orthonormal basis $\{\varphi_i\}_{i=1}^d$ of $\mathbb{C}^d$ such that $\{\varphi_i\}_{i=1}^d$ and $\{U\varphi_i\}_{i=1}^d$ are **mutually unbiased**;
3. there exists an orthonormal basis $\{\varphi_i\}_{i=1}^d$ of $\mathbb{C}^d$ in which $\sqrt{d}\,U$ is represented by a **complex Hadamard matrix**, i.e.,
$$|\langle\varphi_i|U|\varphi_j\rangle| = \frac{1}{\sqrt{d}} \quad \text{for each} \quad i,j = 1,\dots,d.$$

Simple necessary condition: U chaotic $\implies |\mathrm{tr}\,U| \le \sqrt{d}$.

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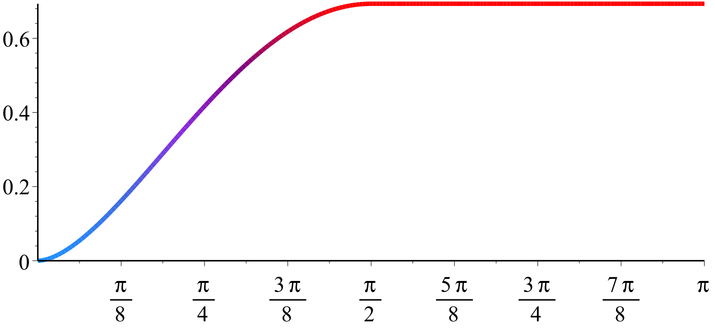
$$\det U_\theta = 1$$

$$\operatorname{tr} U_\theta = 2\cos\frac{\theta}{2} \in [-2, 2]$$

$$H^{\text{dyn}}(U_\theta) = \ln 2 \iff |\operatorname{tr} U_\theta| \le \sqrt{2}$$



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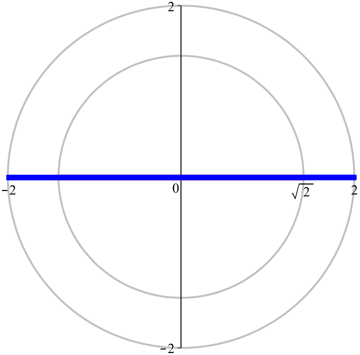

Traces of:

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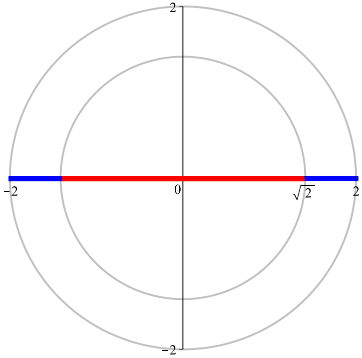

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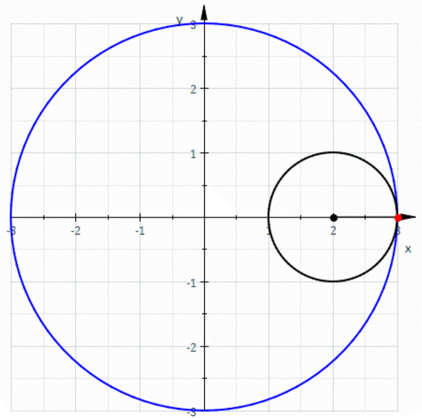
Traces of special unitaries

Theorem (Charzyński et al. 2005). All possible traces of **special** unitary matrices of size $d \times d$ fill in **the d-hypocycloid** with one cusp at $(d, 0)$.



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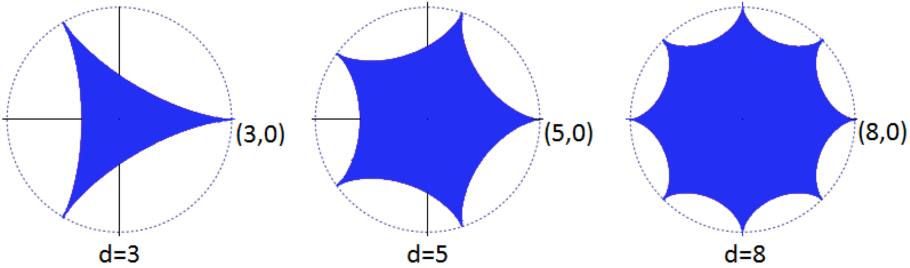
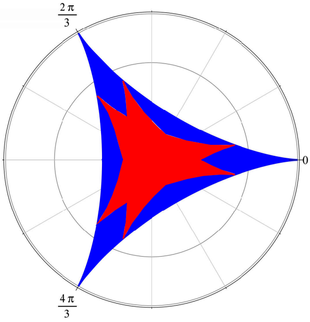


Qutrits

A special 3×3 unitary U is chaotic $\Longrightarrow$

$$\operatorname{tr} U \in \left(\frac{1}{\sqrt{3}} e^{\pi i/18}\,\text{Hypocycloid}(3) \;\cup\; \frac{1}{\sqrt{3}} e^{-\pi i/18}\,\text{Hypocycloid}(3) \right)$$



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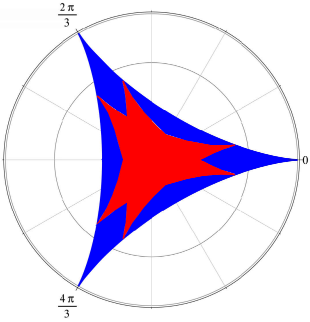

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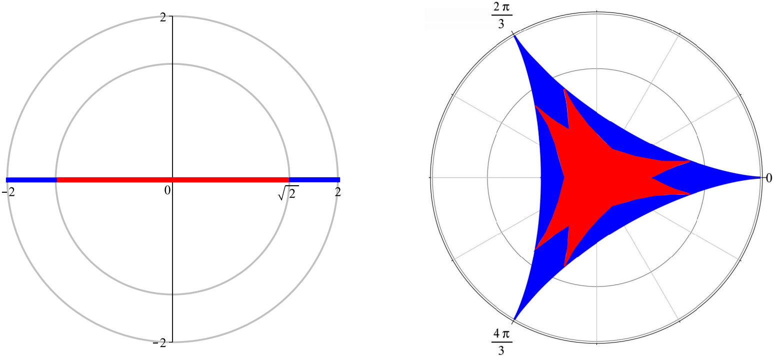
Necessary trace condition for chaoticity

In every dimension d: traces of special chaotic unitaries fill in the union of d-hypocycloids that are **rescaled** (by $1/\sqrt{d}$) and **rotated**. Rotation factors are related to the equivalency classes of **complex Hadamard matrices**.



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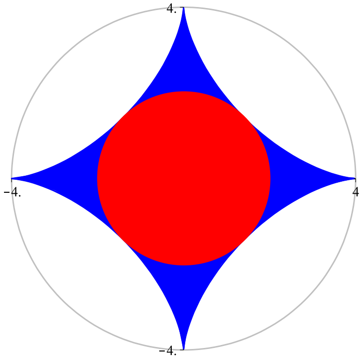

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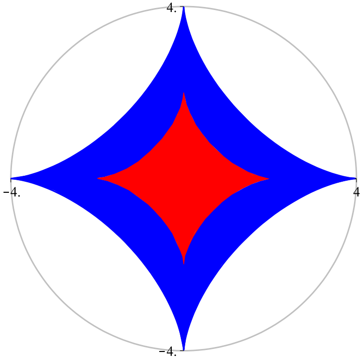

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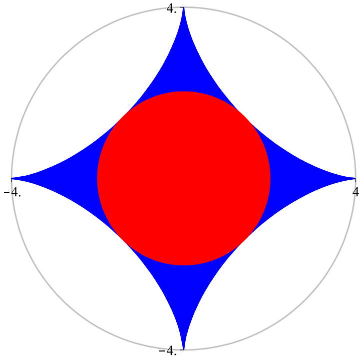

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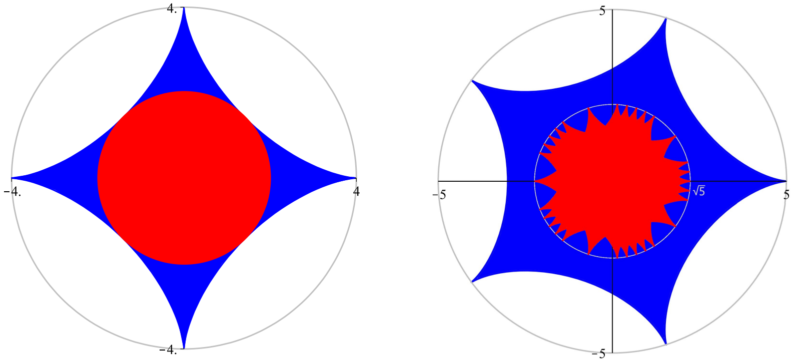


All possible traces of: special unitaries, special chaotic unitaries

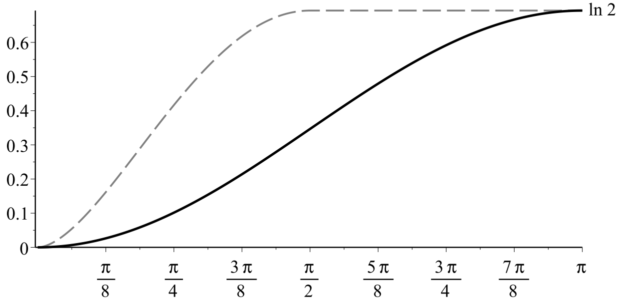
Multiple qubits

$U_\theta := \mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta/2}, \mathrm{e}^{-\mathrm{i}\theta/2})$ for $\theta \in [0, \pi]$

For every $n \in \mathbb{N}$: $U_\theta^{\otimes n}$ is chaotic $\iff$ U_θ is chaotic

For every $n \in \mathbb{N}$: $U_\pi \otimes \mathbb{I}_2^{\otimes n-1}$ is chaotic

$$\lim_{n\to\infty} \frac{1}{n} H^{\mathrm{dyn}}(U_\theta \otimes \mathbb{I}_2^{\otimes n-1}) = \frac{1-\cos\theta}{2} \ln 2$$



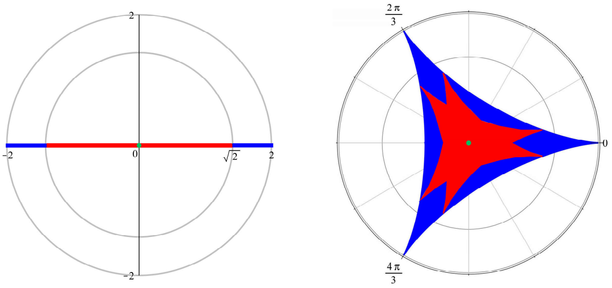
$U \in \mathcal{U}(\mathbb{C}^d)$ is called **stubbornly chaotic** if $U \otimes \mathbb{I}_d^{\otimes n}$ is chaotic for each $n \in \mathbb{N}$.

Necessary condition: U is stubbornly chaotic $\Rightarrow$ $\operatorname{tr} U = 0$.

In dim 2 and 3 this condition is also sufficient.

In dim 2 and 3 there are unique (up to a phase) stubbornly chaotic unitaries:

$$\operatorname{diag}(1, -1) \quad \text{and} \quad \operatorname{diag}(1, e^{2\pi i/3}, e^{4\pi i/3})$$



Traces of special operators: unitary, chaotic, stubbornly chaotic

non-projective measurements: two sources of randomness

Quantum dynamical entropy of U with respect to any rank-1 POVM Π:

$$\underbrace{H_{dyn}(U,\Pi)}_{\substack{\text{entropy due to}\\ \text{unitary dynamics}}} := \underbrace{H(U,\Pi)}_{\substack{\text{total}\\ \text{entropy}}} - \underbrace{H(\mathbb{I},\Pi)}_{\substack{\text{entropy due to}\\ \text{measurement}}}$$

Quantum dynamical entropy of U (independent of measurement):

$$H^{dyn}_{POVM}(U) := \sup\{H_{dyn}(U,\Pi) : \Pi \text{ is a rank-1 POVM}\}$$

1. $-\ln d < H_{dyn}(U,\Pi) \leq \ln d$
2. $0 \leq H^{dyn}_{PVM}(U) \leq H^{dyn}_{POVM}(U) \leq \ln d$;
3. $H^{\rm dyn}_{POVM}(U) = \ln d$ iff $H^{\rm dyn}_{PVM}(U) = \ln d$
4. In dim 2 we have $H^{\rm dyn}_{POVM}(U) = H^{\rm dyn}_{PVM}(U)$ for every U.

References

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