Quantum advantage in simulating stochastic processes

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Outline

1. Motivating example
2. Classical and quantum embeddability
3. Quantum advantages
   - Power of memoryless quantum dynamics
   - Space-time trade-off improvements
   - Memory advantages in control
4. Outlook

arXiv:2005.02403 [pdf, other]

Quantum advantage in simulating stochastic processes

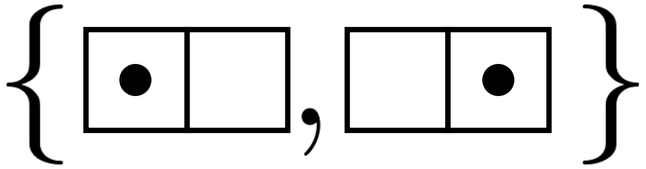
Kamil Korzekwa, Matteo Lostaglio

Comments: 20 pages, 10 figures. Comments welcome

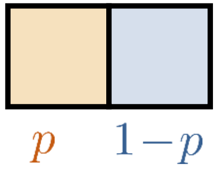
Subjects: **Quantum Physics (quant-ph)**

Motivating example

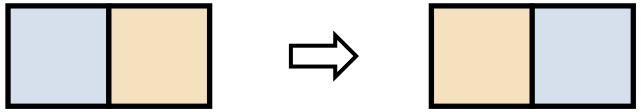
Two-level system



Classical probabilistic description of the system's state



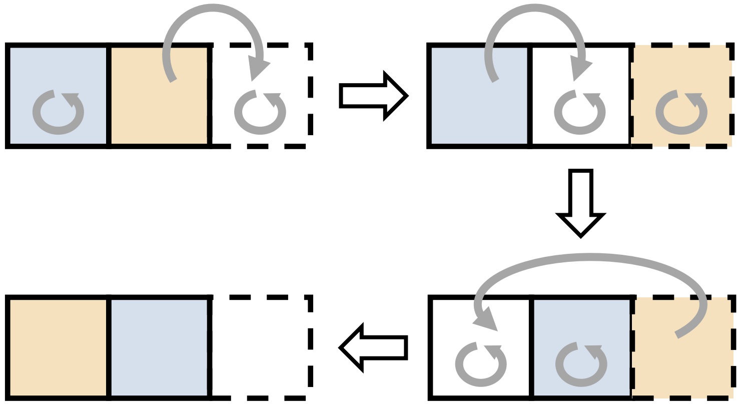
Flip operation



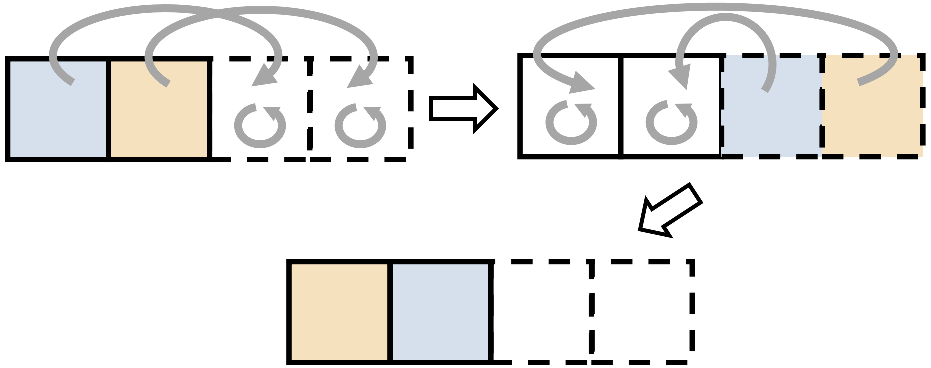
Markovian time-continuous flip operation?

Impossible with classical probabilities

1 memory state, 3 time-steps

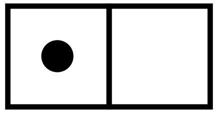
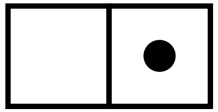


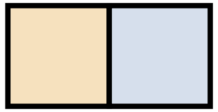
2 memory state, 2 time-steps

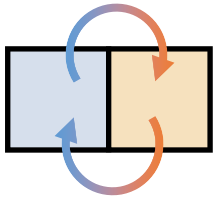
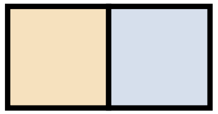


Motivating example

Markovian time-continuous flip operation possible via quantum evolution

 $\rightarrow |0\rangle$  $\rightarrow |1\rangle$

 $\rightarrow p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|$

 $U(t) = e^{i\sigma_x t}$ $\Rightarrow$ 

$U(0)\rho U^\dagger(0) = \rho$ $\qquad$ $U(\frac{\pi}{2})\rho U^\dagger(\frac{\pi}{2}) = \sigma_x \rho \sigma_x$

Questions:

1. Can we simulate classical processes requiring memory with quantum memoryless dynamics?
2. Beyond yes/no answer: can we get some quantum memory advantages?
3. Can we employ those advantages in control theory?

Classical and quantum embeddability

Stochastic evolution of a d-level system: $P = \begin{pmatrix} P_{11} & P_{12} & \dots & P_{1d} \\ P_{21} & P_{22} & \dots & P_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ P_{d1} & P_{d2} & \dots & P_{dd} \end{pmatrix}, \quad P_{ij} \geq 0, \quad \sum_i P_{ij} = 1$

Embeddable stochastic matrices: $\frac{d}{dt}P(t) = L(t)P(t), \quad P(0) = \mathbb{1}$

Generator of the evolution: $L_{ij} \geq 0$ for $i \neq j$, $\quad \sum_i L_{ij} = 0$

For time-independent generator it means: $P(t) = e^{Lt}$

$$P(0) = \mathbb{1} \xrightarrow{t \in (0,1)} P(1) = e^{L}$$

Classical and quantum embeddability

Embeddability problem introduced in 1937 by Elfving

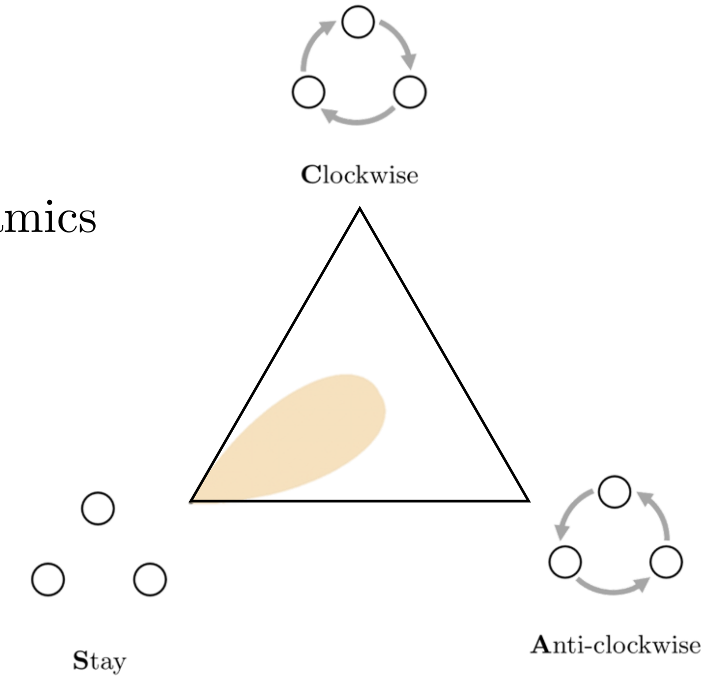
G. Elfving, *Zur theorie der Markoffschen ketten*, Acta Soc. Sci. Fennicae, n. Ser. A2 8, 1–17 (1937).

After more than 80 years still unsolved for d>3!

Known necessary conditions

G. Goodman, *An intrinsic time for non-stationary finite Markov chains*, Probab. Theory Relat. Fields 16, 165–180 (1970).

$$\prod_i P_{ii} \geq \det P \geq 0$$



Two-level dynamics
(total probability to stay larger than to change)

$$P = \begin{pmatrix} a & 1-b \\ 1-a & b \end{pmatrix},$$

$$a + b \geq 1$$

Three-level circulant dynamics

$$P = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix},$$

Classical and quantum embeddability

Stochastic evolution generated by a quantum channel: $P_{ij} = \langle i|\mathcal{E}(|j\rangle\langle j|)|i\rangle$, with $\{|i\rangle\}$ denoting the distinguished basis

Markovian quantum channels: $\frac{d}{dt}\mathcal{E}(t) = \mathcal{L}(t)\mathcal{P}(t)$, $\mathcal{E}(0) = \mathcal{I}$

Generator of the evolution: $\mathcal{L}(\cdot) = -i[H,\cdot] + \Phi(\cdot) - \frac{1}{2}\{\Phi^*(\mathbb{1}),\cdot\}$

Definition 1 (Quantum embeddable stochastic matrix)**.** A stochastic matrix P is *quantum embeddable* if

$$P_{ij} = \langle i|\mathcal{E}\left(|j\rangle\langle j|\right)|i\rangle,$$

where $\mathcal{E}$ is a Markovian quantum channel.

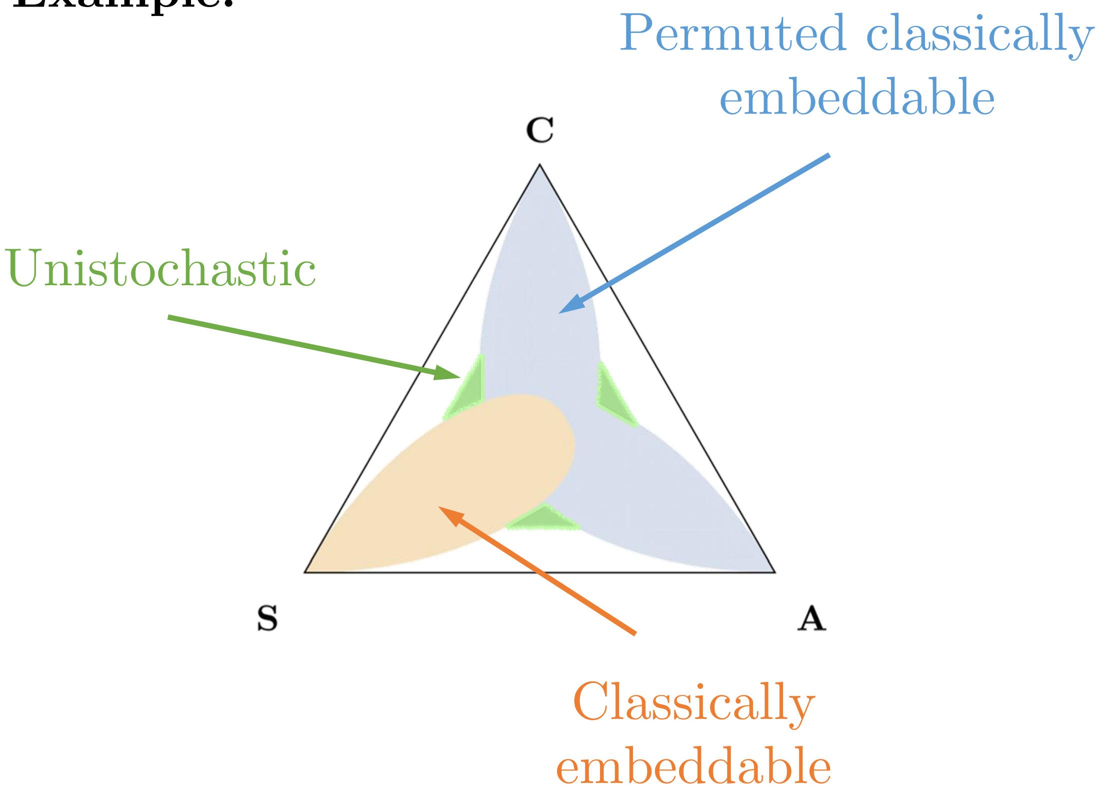
Power of memoryless quantum dynamics

Unistochastic matrix: $P_{ij} = |\langle j|U|i\rangle|^2$, where U is unitary

Results/observations:

- All classically embeddable matrices are also quantum embeddable.
- All unistochastic matrices are quantum embeddable.
- A product of quantum embeddable matrices is also quantum embeddable.
- All 2 x 2 matrices are quantum embeddable.

Example:



Space-time trade-off improvements

D. Wolpert, *et al.*, *A space–time tradeoff for implementing a function with master equation dynamics,* Nat. Commun. 10, 1–9 (2019)

Definition 2 (Space cost). The *space cost* of a $d \times d$ stochastic matrix P, denoted $C_{\text{space}}(P)$, is the minimum m such that the $(d+m)\times(d+m)$ embeddable matrix Q implements P.

Definition 3 (Time cost). The *time cost* $C_{\text{time}}(P, m)$ of a $d \times d$ stochastic matrix P, while allowing for m memory states, is the minimum number τ of one-step stochastic matrices $T^{(i)}$ of dimension $(d+m) \times (d+m)$ such that $Q = T^{(\tau)} \cdots T^{(1)}$ implements P.

One-step stochastic matrix: embeddable stochastic matrix with a time-independent generator*

E.g. $Q = e^{L^{(n)} t_n} \cdots e^{L^{(1)} t_1}$ is an n-step process

*Actually, in the classical case Wolpert *et al.* consider a broader notion of a one-step process.

Space-time trade-off improvements

D. Wolpert, *et al.*, *A space–time tradeoff for implementing a function with master equation dynamics,* Nat. Commun. 10, 1–9 (2019)

Consider a class of $\{0,1\}$-valued $d \times d$ stochastic matrices

Such a matrix P_f is defined by a function $f : \mathbb{Z}_d \to \mathbb{Z}_d$.

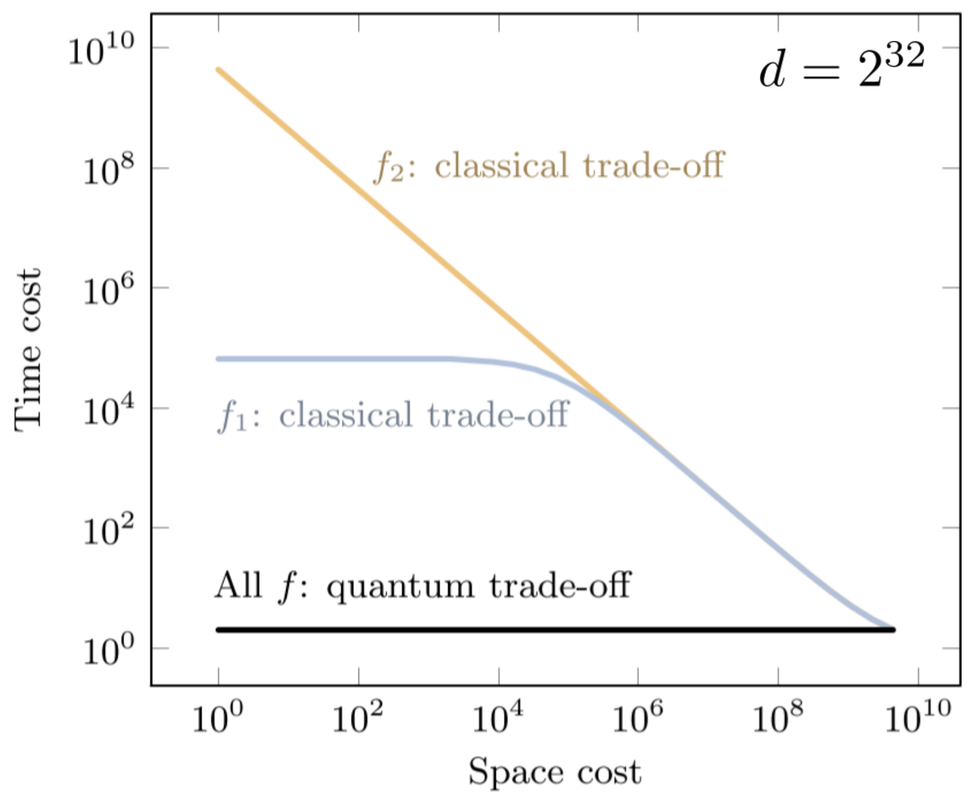
Classical trade-off:

$$C_{\text{time}}(P_f, m) \geq \left\lceil \frac{m + d - \text{fix}(f)}{m + d - |\text{img}(f)|} \right\rceil,$$

Our result

For any $m \geq 0$ and any function f we have:

$$Q_{\text{time}}(P_f, m) \leq 2.$$



$f_1(i) = i \oplus 1, \; f_2(i) = \min\{i + 2^{s/2}, 2^s - 1\}$

Memory advantages in control

Instead on procesess let us focus on state transformations: $\boldsymbol{p} \to \boldsymbol{q}$

Q: Does Markovianity alone restrict our power to perform certain state transformations?
A: No. Simply choose a Markovian process with a unique fixed point $\boldsymbol{q}$.

Realistically: fixed point of the evolution is constrained, e.g., to be the thermal state: $\gamma_k := \frac{1}{Z} e^{-\beta E_k}, \quad Z := \sum_{k=1}^{d} e^{-\beta E_k}$

ACCESSIBILITY REGIONS	Classical	Quantum
With memory	$P\boldsymbol{p} = \boldsymbol{q}, \quad P\boldsymbol{\gamma} = \boldsymbol{\gamma}$	$\mathcal{E}(\rho_{\boldsymbol{p}}) = \rho_{\boldsymbol{q}}, \quad \mathcal{E}(\rho_{\boldsymbol{\gamma}}) = \rho_{\boldsymbol{\gamma}}$
Without memory	$P\boldsymbol{p} = \boldsymbol{q}, \quad P\boldsymbol{\gamma} = \boldsymbol{\gamma}$ P Markovian	$\mathcal{E}(\rho_{\boldsymbol{p}}) = \rho_{\boldsymbol{q}}, \quad \mathcal{E}(\rho_{\boldsymbol{\gamma}}) = \rho_{\boldsymbol{\gamma}}$ $\mathcal{E}$ Markovian

Where: $\rho_{\boldsymbol{p}} := \sum_i p_i |i\rangle\langle i|$

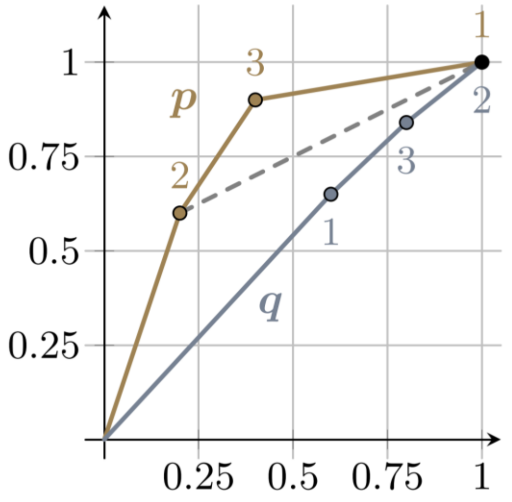
Memory advantages in control

Classical accessibility regions

With memory: known conditions specified by thermo-majorization

$$\boldsymbol{p} \to \boldsymbol{q} \iff \boldsymbol{p} \succ_\gamma \boldsymbol{q}$$

(encodes, i.a., the non-increasing of free energy)



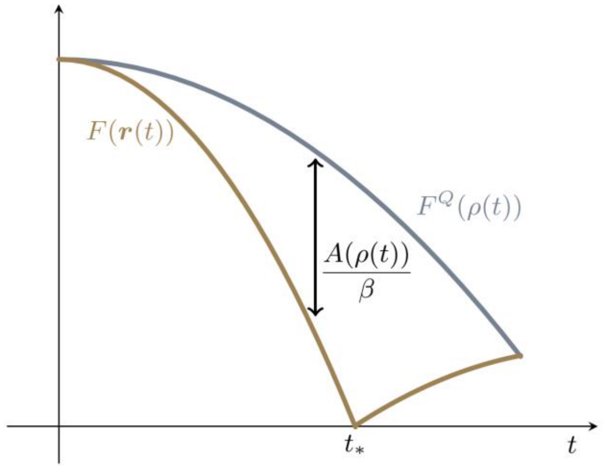
Without memory: we introduce & characterize the new notion of **Markovian thermo-majorization**

$\boldsymbol{p} \ggcurly_\gamma \boldsymbol{q} \iff$ there exists $\boldsymbol{r}(t)$ such that:

1. $\boldsymbol{r}(0) = \boldsymbol{p}$,
2. $\forall\, t_1, t_2 \in [0, t_f) : \quad t_1 \leq t_2 \Rightarrow \boldsymbol{r}(t_1) \succ_\gamma \boldsymbol{r}(t_2)$,
3. $\boldsymbol{r}(t_f) = \boldsymbol{q}$.

Memory advantages in control

Quantum accessibility regions

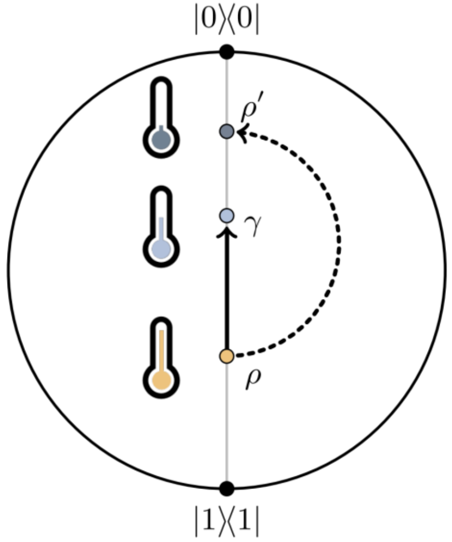


Maximal quantum advantage for uniform fixed points:

All states that can be classicaly (with memory) achieved from $\boldsymbol{p}$ can also be achieved by quantum evolution without memory

Maximal quantum advantage for general fixed points for two-level systems:

All states* that can be classicaly (with memory) achieved from $\boldsymbol{p}$ can also be achieved by quantum evolution without memory



*Actually even stronger results holds: the set of all quantum states that can be achieved via quanutm channels with a given fixed point, can be achieved in a Markovian way.

Outlook

- Derive more stringent conditions for quantum embeddability
- Extend space-time trade-off analysis beyond $\{0,1\}$-valued stochastic processes
- Find physical realisations of qubit Lindbladians providing maximal quantum advantage
- Investigate practical advantages for near-term quantum devices
- Establish a stronger link between stochastic thermodynamics and resource theories

Thank you!

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