

Quantum advantage in simulating stochastic processes

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TEAM-NET

Outline

1. Motivating example
2. Classical and quantum embeddability
3. Quantum advantages
 - Power of memoryless quantum dynamics
 - Space-time trade-off improvements
 - Memory advantages in control
4. Outlook



[arXiv:2005.02403](#) [pdf, other]

Quantum advantage in simulating stochastic processes

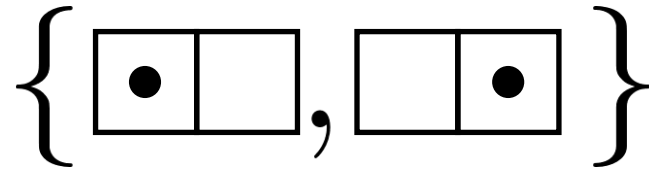
[Kamil Korzekwa](#), [Matteo Lostaglio](#)

Comments: 20 pages, 10 figures. Comments welcome

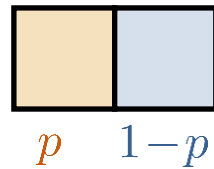
Subjects: **Quantum Physics (quant-ph)**

Motivating example

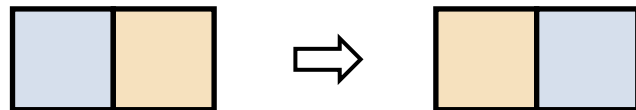
Two-level system



Classical probabilistic description
of the system's state



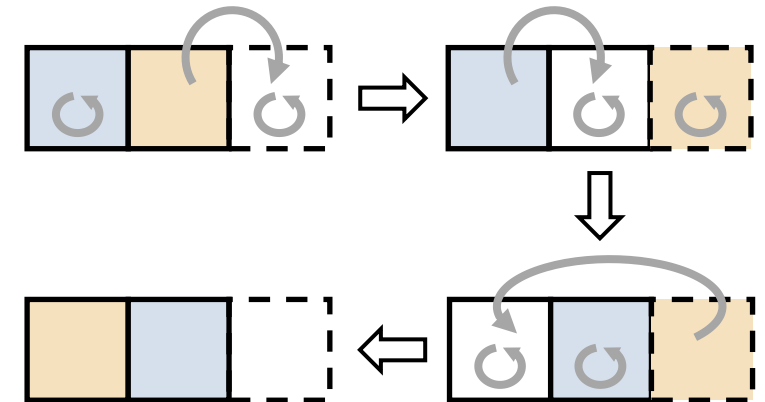
Flip operation



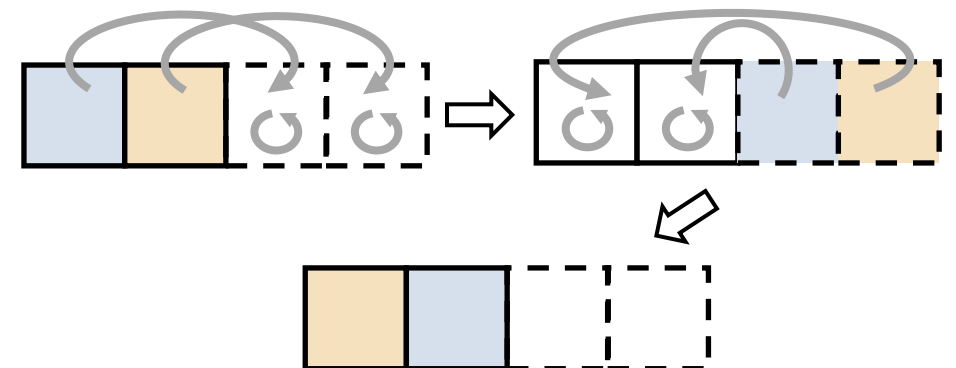
Markovian time-continuous flip operation?

Impossible with classical probabilities

1 memory state, 3 time-steps



2 memory state, 2 time-steps



Motivating example

Markovian time-continuous flip operation possible via quantum evolution

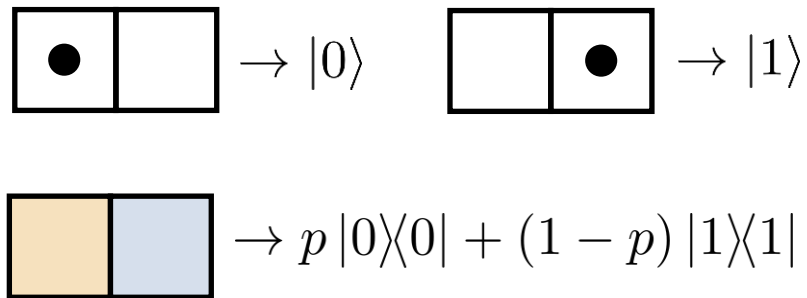


Diagram illustrating the quantum evolution operation $U(t) = e^{i\sigma_x t}$ and its effect on a state ρ :

- The operation $U(t) = e^{i\sigma_x t}$ is shown as a box with two halves (left light blue, right orange) and two curved arrows (one blue, one orange) indicating a swap between the halves.
- The operation is represented by the equation $U(t) = e^{i\sigma_x t}$.
- The operation is represented by the equation $U(0)\rho U^\dagger(0) = \rho$.
- The operation is represented by the equation $U(\frac{\pi}{2})\rho U^\dagger(\frac{\pi}{2}) = \sigma_x \rho \sigma_x$.

Questions:

1. Can we simulate classical processes requiring memory with quantum memoryless dynamics?
2. Beyond yes/no answer: can we get some quantum memory advantages?
3. Can we employ those advantages in control theory?

Classical and quantum embeddability

Stochastic evolution of a d-level system:

$$P = \begin{pmatrix} P_{11} & P_{12} & \dots & P_{1d} \\ P_{21} & P_{22} & \dots & P_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ P_{d1} & P_{d2} & \dots & P_{dd} \end{pmatrix}, \quad P_{ij} \geq 0, \quad \sum_i P_{ij} = 1$$

Embeddable stochastic matrices:

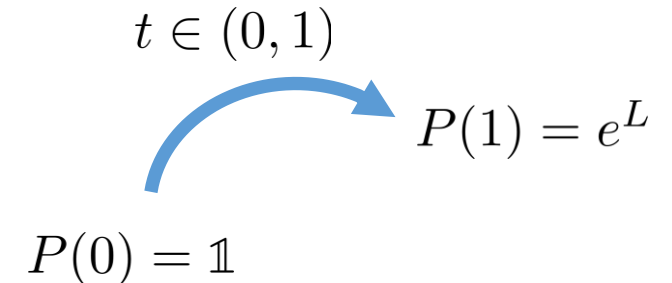
$$\frac{d}{dt}P(t) = L(t)P(t), \quad P(0) = \mathbb{1}$$

Generator of the evolution:

$$L_{ij} \geq 0 \text{ for } i \neq j, \quad \sum_i L_{ij} = 0$$

For time-independent generator it means:

$$P(t) = e^{Lt}$$


$$P(0) = \mathbb{1} \xrightarrow{t \in (0, 1)} P(1) = e^L$$

Classical and quantum embeddability

Embeddability problem introduced in 1937 by Elfving

G. Elfving, *Zur theorie der Markoffschen ketten*, Acta Soc. Sci. Fennicae, n. Ser. A2 8, 1–17 (1937).

After more than 80 years still unsolved for $d > 3$!

Known necessary conditions

G. Goodman, *An intrinsic time for non-stationary finite Markov chains*, Probab. Theory Relat. Fields 16, 165–180 (1970).

$$\prod_i P_{ii} \geq \det P \geq 0$$

Two-level dynamics

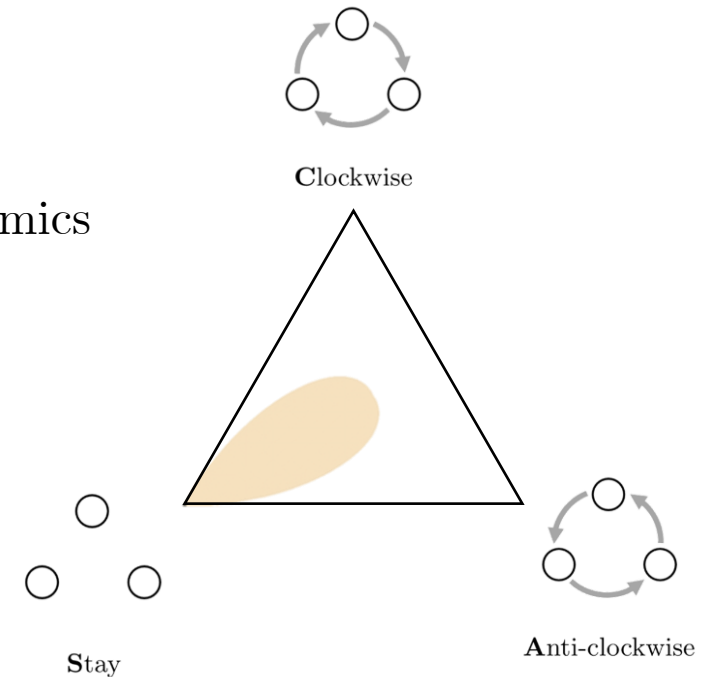
(total probability to stay larger than to change)

$$P = \begin{pmatrix} a & 1-b \\ 1-a & b \end{pmatrix},$$

$$a + b \geq 1$$

Three-level circulant dynamics

$$P = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix},$$



Classical and quantum embeddability

Stochastic evolution generated by a quantum channel:

$$P_{ij} = \langle i | \mathcal{E}(|j\rangle\langle j|) | i \rangle, \quad \text{with } \{|i\rangle\} \text{ denoting the distinguished basis}$$

Markovian quantum channels:

$$\frac{d}{dt} \mathcal{E}(t) = \mathcal{L}(t) \mathcal{P}(t), \quad \mathcal{E}(0) = \mathcal{I}$$

Generator of the evolution:

$$\mathcal{L}(\cdot) = -i[H, \cdot] + \Phi(\cdot) - \frac{1}{2} \{ \Phi^*(\mathbb{1}), \cdot \}$$

Definition 1 (Quantum embeddable stochastic matrix). A stochastic matrix P is *quantum embeddable* if

$$P_{ij} = \langle i | \mathcal{E}(|j\rangle\langle j|) | i \rangle,$$

where \mathcal{E} is a Markovian quantum channel.

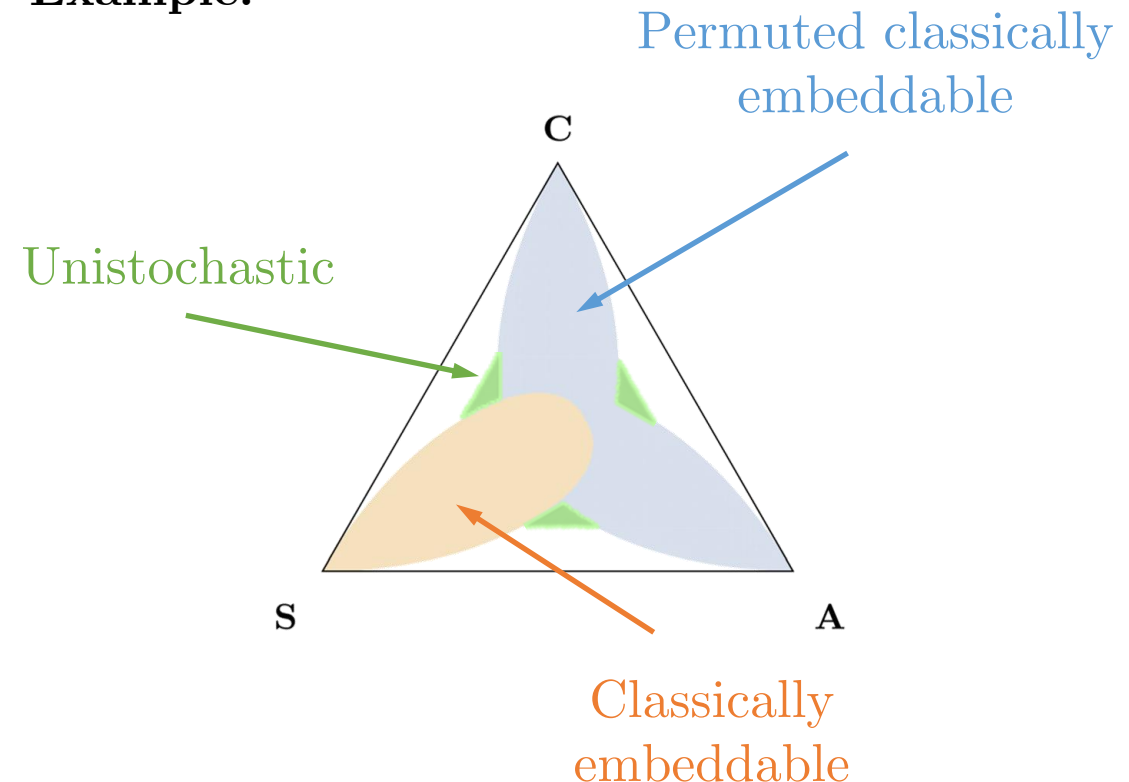
Power of memoryless quantum dynamics

Unistochastic matrix: $P_{ij} = |\langle j|U|i\rangle|^2$, where U is unitary

Results/observations:

- All classically embeddable matrices are also quantum embeddable.
- All unistochastic matrices are quantum embeddable.
- A product of quantum embeddable matrices is also quantum embeddable.
- All 2 x 2 matrices are quantum embeddable.

Example:



Space-time trade-off improvements

D. Wolpert, *et al.*, *A space-time tradeoff for implementing a function with master equation dynamics*, Nat. Commun. 10, 1–9 (2019)

Definition 2 (Space cost). The *space cost* of a $d \times d$ stochastic matrix P , denoted $C_{\text{space}}(P)$, is the minimum m such that the $(d+m) \times (d+m)$ embeddable matrix Q implements P .

Definition 3 (Time cost). The *time cost* $C_{\text{time}}(P, m)$ of a $d \times d$ stochastic matrix P , while allowing for m memory states, is the minimum number τ of one-step stochastic matrices $T^{(i)}$ of dimension $(d+m) \times (d+m)$ such that $Q = T^{(\tau)} \dots T^{(1)}$ implements P .

One-step stochastic matrix: embeddable stochastic matrix with a time-independent generator*

E.g. $Q = e^{L^{(n)}t_n} \dots e^{L^{(1)}t_1}$ is an n -step process

*Actually, in the classical case Wolpert *et al.* consider a broader notion of a one-step process.

Space-time trade-off improvements

D. Wolpert, *et al.*, *A space-time tradeoff for implementing a function with master equation dynamics*, Nat. Commun. 10, 1–9 (2019)

Consider a class of $\{0, 1\}$ -valued $d \times d$ stochastic matrices

Such a matrix P_f is defined by a function $f : \mathbb{Z}_d \rightarrow \mathbb{Z}_d$.

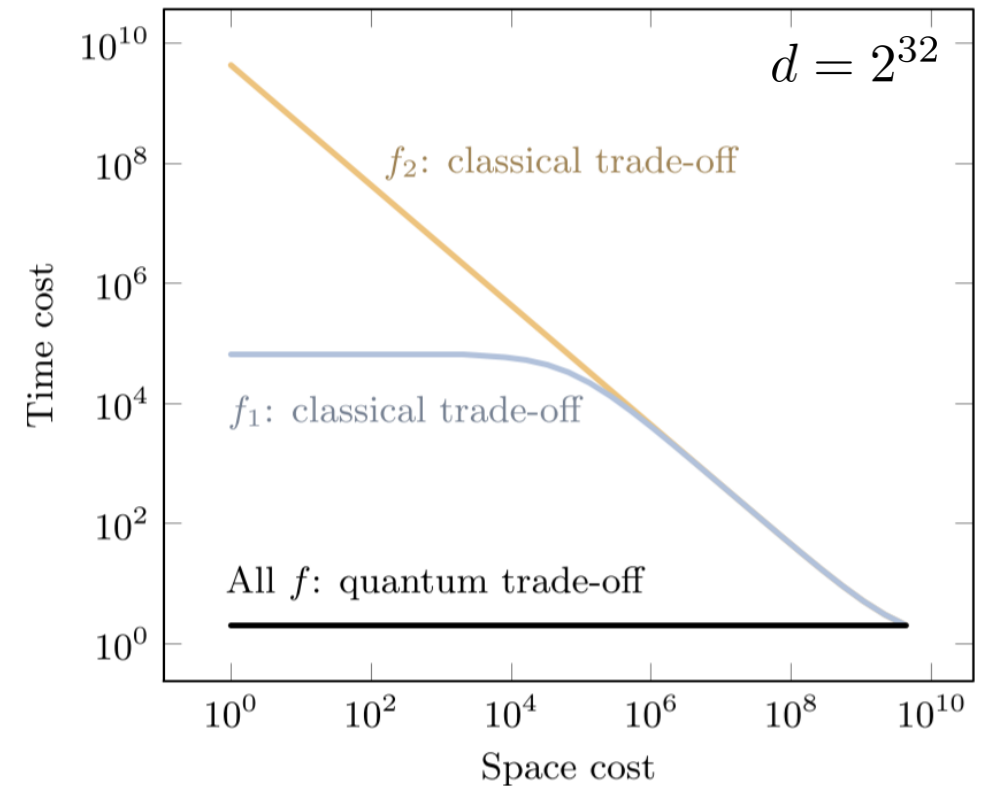
Classical trade-off:

$$C_{\text{time}}(P_f, m) \geq \left\lceil \frac{m + d - \text{fix}(f)}{m + d - |\text{img}(f)|} \right\rceil,$$

Our result

For any $m \geq 0$ and any function f we have:

$$Q_{\text{time}}(P_f, m) \leq 2.$$



$$f_1(i) = i \oplus 1, \quad f_2(i) = \min\{i + 2^{s/2}, 2^s - 1\}$$

Memory advantages in control

Instead on procesess let us focus on state transformations: $\mathbf{p} \rightarrow \mathbf{q}$

Q: Does Markovianity alone restrict our power to perform certain state transformations?

A: No. Simply choose a Markovian process with a unique fixed point \mathbf{q} .

Realistically: fixed point of the evolution is constrained,
e.g., to be the thermal state:

$$\gamma_k := \frac{1}{Z} e^{-\beta E_k}, \quad Z := \sum_{k=1}^d e^{-\beta E_k}$$

ACCESSIBILITY REGIONS	Classical	Quantum
With memory	$P\mathbf{p} = \mathbf{q}, \quad P\gamma = \gamma$	$\mathcal{E}(\rho_{\mathbf{p}}) = \rho_{\mathbf{q}}, \quad \mathcal{E}(\rho_{\gamma}) = \rho_{\gamma}$
Without memory	$P\mathbf{p} = \mathbf{q}, \quad P\gamma = \gamma$ P Markovian	$\mathcal{E}(\rho_{\mathbf{p}}) = \rho_{\mathbf{q}}, \quad \mathcal{E}(\rho_{\gamma}) = \rho_{\gamma}$ \mathcal{E} Markovian

Where: $\rho_{\mathbf{p}} := \sum_i p_i |i\rangle\langle i|$

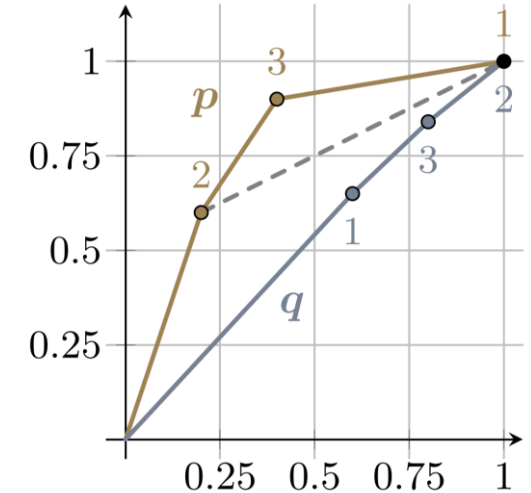
Memory advantages in control

Classical accessibility regions

With memory: known conditions specified by thermo-majorization

$$\mathbf{p} \rightarrow \mathbf{q} \iff \mathbf{p} \succ_{\gamma} \mathbf{q}$$

(encodes, i.a., the non-increasing of free energy)



Without memory: we introduce & characterize the new notion of **Markovian thermo-majorization**

$$1. \mathbf{r}(0) = \mathbf{p},$$

$$\mathbf{p} \gg_{\gamma} \mathbf{q} \iff \text{there exists } \mathbf{r}(t) \text{ such that: } 2. \forall t_1, t_2 \in [0, t_f) : t_1 \leq t_2 \Rightarrow \mathbf{r}(t_1) \succ_{\gamma} \mathbf{r}(t_2),$$

$$3. \mathbf{r}(t_f) = \mathbf{q}.$$

Memory advantages in control

Quantum accessibility regions

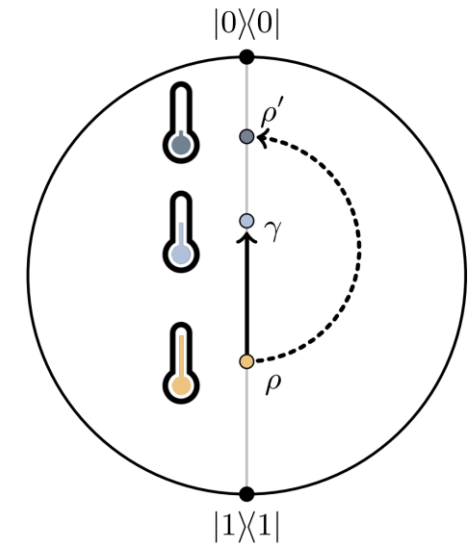
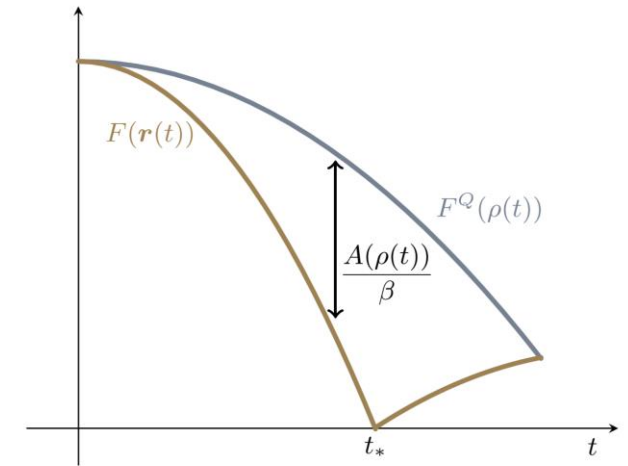
Maximal quantum advantage for uniform fixed points:

All states that can be classically (with memory) achieved from \mathbf{p} can also be achieved by quantum evolution without memory

Maximal quantum advantage for general fixed points for two-level systems:

All states* that can be classically (with memory) achieved from \mathbf{p} can also be achieved by quantum evolution without memory

*Actually even stronger results holds: the set of all quantum states that can be achieved via quantum channels with a given fixed point, can be achieved in a Markovian way.



Outlook

- Derive more stringent conditions for quantum embeddability
- Extend space-time trade-off analysis beyond $\{0,1\}$ -valued stochastic processes
- Find physical realisations of qubit Lindbladians providing maximal quantum advantage
- Investigate practical advantages for near-term quantum devices
- Establish a stronger link between stochastic thermodynamics and resource theories

Thank you!

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