# Evolution of open quantum systems governed by unitarily covariant quantum channels

**Author:** Dr. Katarzyna Siudzińska

**Advisor:** Prof. Dariusz Chruściński

Institute of Physics, Department of Mathematical Physics
Nicolaus Copernicus University in Toruń, Poland

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NICOLAUS COPERNICUS UNIVERSITY IN TORUŃ
Faculty of Physics, Astronomy and Informatics

# Scope of my PhD thesis

**Analyzing the evolution of open quantum systems** described by unitarily covariant quantum channels, including

1. application of the group theory to construct covariant channels,

$$U\Lambda[X]U^\dagger = \Lambda[UXU^\dagger];$$

2. description of quantum evolution using local and non-local master equations,

$$\dot{\Lambda}(t) = \mathcal{L}(t)\Lambda(t), \qquad \dot{\Lambda}(t) = \int_0^t K(t-\tau)\Lambda(\tau)\,\mathrm{d}\tau;$$

3. determination of Markovianity or non-Markovianity properties of quantum evolution through divisibility of dynamical maps,

$$\Lambda(t) = V(t,\tau)\Lambda(\tau), \quad \tau < t.$$

# Unitarily covariant channels

**Types of channels considered:**

- Pauli channels (bistochastic evolution of a qubit)

$$\Lambda[X] = \sum_{\alpha=0}^{3} p_\alpha \sigma_\alpha X \sigma_\alpha; \tag{1}$$

- Weyl channels (mixed unitary evolution)

$$\Lambda[X] = \sum_{k,l=0}^{d-1} p_{kl} W_{kl} X W_{kl}^\dagger; \tag{2}$$

- generalized Pauli channels

$$\Lambda[X] = p_0 X + \frac{1}{d-1} \sum_{\alpha=1}^{d+1} p_\alpha \sum_{k=1}^{d-1} U_\alpha^k X U_\alpha^{k\dagger}. \tag{3}$$

*M. Nathanson and M. B. Ruskai, J. Phys. A: Math. Theor.* **40**, *8171 (2007).*

# Unitarily covariant channels

A quantum channel $\Lambda$ is covariant with respect to a unitary representation $U$ of a group $G$ if

$$\Lambda[U(g)XU^\dagger(g)] = U(g)\Lambda[X]U^\dagger(g) \tag{4}$$

for every group element $g \in G$ and Hermitian operator $X$.

## Theorem 1

*Let $G$ be a group generated by*

$$W_{kl} = \sum_{m=0}^{d-1} \omega^{km} |m + l \mod d\rangle\langle m|, \qquad \omega = e^{2\pi i/d}. \tag{5}$$

*A quantum channel covariant with respect to a $d$-dimensional irreducible unitary representation of the group $G$ reads*

$$\Lambda[\rho] = \sum_{k,l=0}^{d-1} p_{kl} W_{kl} \rho W_{kl}^\dagger. \tag{6}$$

*N. Datta, M. Fukuda, and A. S. Holevo, Quantum Inf. Process.* **5, 179-207 (2006).**

# Unitarily covariant channels

## Theorem 2

*If $d$ is a prime number, then*

$$\Lambda = p_0 \mathbb{1} + \frac{1}{d-1} \sum_{\alpha=1}^{d+1} p_\alpha \mathbb{U}_\alpha \tag{7}$$

*is the Weyl channel covariant with respect to* **every** *$d$-dimensional irreducible unitary representation of the group $G$, where*

$$\mathbb{U}_\alpha[\rho] = \sum_{k=1}^{d-1} U_\alpha^k \rho U_\alpha^{k\dagger}, \qquad U_\alpha = \sum_{l=0}^{d-1} \omega^l P_l^{(\alpha)}. \tag{8}$$

$P_k^{(\alpha)}$ are rank-1 projectors onto the mutually unbiased bases vectors $\{\psi_0^{(\alpha)}, \ldots, \psi_{d-1}^{(\alpha)}\}$, which satisfy

$$|\langle \psi_k^{(\alpha)} | \psi_l^{(\beta)} \rangle|^2 = \begin{cases} \delta_{kl}, & \alpha = \beta, \\ \frac{1}{d}, & \alpha \neq \beta. \end{cases} \tag{9}$$

# Evolution of open quantum systems

A time-evolution of a quantum system is provided by a **dynamical map** $\Lambda(t)$, $t \geq 0$, where $\Lambda(0) = \mathbb{1}$.

The simplest evolution equation for the density operator $\rho \mapsto \rho(t) = \Lambda(t)[\rho]$ is

$$\dot{\Lambda}(t) = \mathcal{L}\Lambda(t) \tag{10}$$

with the Gorini-Kossakowski-Sudarshan-Lindblad generator

$$\mathcal{L}[\rho] = -i[H,\rho] + \frac{1}{2}\sum_{\alpha}\gamma_\alpha\left(V_\alpha\rho V_\alpha^\dagger - \frac{1}{2}[V_\alpha^\dagger V_\alpha,\rho]_+\right). \tag{11}$$

**Possible generalizations:**

- the master equation with time-local generators $\mathcal{L}(t)$ of the form (11) but with time-dependent $H(t)$, $V_\alpha(t)$ and $\gamma_\alpha(t)$;
- the Nakajima-Zwanzig equation with a memory kernel $K(t,\tau)$,

$$\dot{\Lambda}(t) = \int_0^t K(t,\tau)\Lambda(\tau)\,\mathrm{d}\tau. \tag{12}$$

# Evolution of open quantum systems

Consider the generalized Pauli channel $\Lambda(t)$ generated by

$$\mathcal{L}(t) = \frac{1}{d} \sum_{\alpha=1}^{d+1} \gamma_\alpha(t)[\mathbb{U}_\alpha - (d-1)\mathbb{1}]. \tag{13}$$

Its eigenvalues are given by

$$\lambda_\alpha(t) = \exp\left[\Gamma_0(t) - \Gamma_\alpha(t)\right], \qquad \Lambda(t)[U_\alpha^k] = \lambda_\alpha(t) U_\alpha^k, \tag{14}$$

with $\Gamma_\alpha(t) = \int_0^t \gamma_\alpha(\tau)\,\mathrm{d}\tau$ and $\gamma_0(t) = \sum_{\alpha=1}^{d+1} \gamma_\alpha(t)$.

## Theorem 3

*A generator $\mathcal{L}(t)$ describes a physical evolution if and only if*

$$\sum_{\alpha=1}^{d+1} e^{\Gamma_\alpha(t)} \leq e^{\Gamma_0(t)} + d \min_{\beta>0} e^{\Gamma_\beta(t)}. \tag{15}$$

# Markovianity of quantum evolution

A dynamical map

$$\Lambda(t) = V(t,s)\Lambda(s) \tag{16}$$

is

- **CP-divisible** if and only if $V(t,s)$ is completely positive;
- **P-divisible** if and only if $V(t,s)$ is positive but not completely positive.

**Definition 1**

A **Markovian** evolution is given by a CP-divisible $\Lambda(t)$.

The dynamical map generated by the time-local generator

$$\mathcal{L}(t)[\rho] = -i[H(t),\rho] + \frac{1}{2}\sum_\alpha \gamma_\alpha(t)\left(V_\alpha(t)\rho V_\alpha^\dagger(t) - \frac{1}{2}[V_\alpha^\dagger(t)V_\alpha(t),\rho]_+\right) \tag{17}$$

is CP-divisible if and only if $\gamma_\alpha(t) \geq 0$.

*D. Chruściński and A. Kossakowski, J. Phys. B: At. Mol. Opt. Phys.* **45**, *154002 (2012).*

### Definition 2

An evolution is **weakly non-Markovian** if $\Lambda(t)$ is P-divisible.

*D. Chruściński and S. Maniscalco, Phys. Rev. Lett.* **112**, *120404 (2014).*

### Theorem 4

*If the Weyl channel is P-divisible, then*

$$\frac{\mathrm{d}}{\mathrm{d}t}|\lambda_{kl}(t)| \leq 0. \tag{18}$$

### Theorem 5

*If the generalized Pauli channel is P-divisible, then*

$$\sum_{\alpha \neq \beta} \gamma_\alpha(t) \geq 0. \tag{19}$$

# Markoviantiy of quantum evolution

**Theorem 6**

*An invertible Weyl channel is P-divisible if any sum of $d$ decoherence rates $\gamma_{kl}(t)$ is non-negative.*

**Theorem 7**

*An invertible generalized Pauli channel is P-divisible if*

$$\forall_{\alpha\neq\beta} \quad \gamma_\alpha(t)+(d-1)\gamma_\beta(t)\geq 0. \tag{20}$$

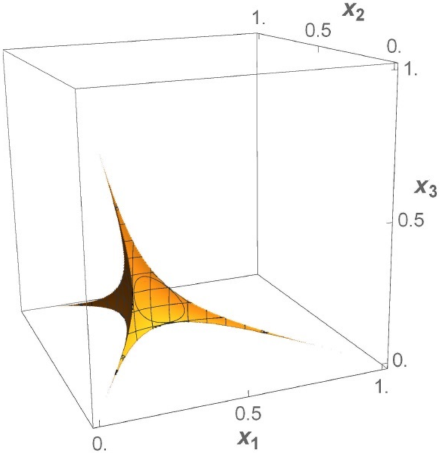
*Less restrictive conditions read*

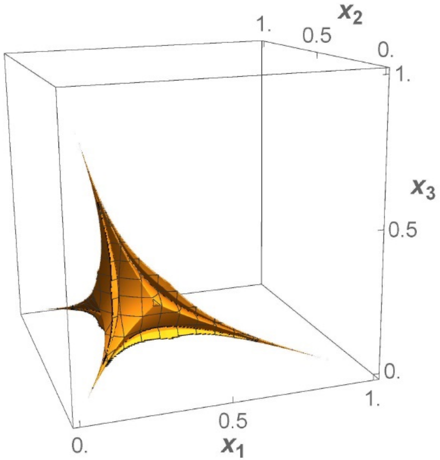
$$\forall_{\gamma_\beta(t)\geq 0} \quad [d-2(k-1)]\gamma_\beta(t)+[d+2(k-1)]\gamma_\alpha(t)\geq 0, \tag{21}$$

*where $k\leq(d+1)/2$.*

# Markovianity of quantum evolution

$$\Lambda(t) = \sum_{\alpha=1}^{4} x_\alpha e^{3t\mathcal{L}_\alpha} = \frac{1}{3}\left[(1+2e^{-3t})\mathbb{1} + (1-e^{-3t})\sum_{\alpha=1}^{4} x_\alpha \mathbb{U}_\alpha\right]$$



# Memory kernels

Consider the evolution given by the generalized Pauli channel that solves the equation

$$\dot{\Lambda}(t) = (K * \Lambda)(t) := \int_0^t K(t-\tau)\Lambda(\tau)\,\mathrm{d}\tau \tag{22}$$

with the memory kernel

$$K(t) = \frac{1}{d}\sum_{\alpha=1}^{d+1} k_\alpha(t)\left[\mathbb{U}_\alpha - (d-1)\mathbb{1}\right]. \tag{23}$$

$K(t)$ has the same eigenvectors as $\Lambda(t)$,

$$\Lambda(t)[U_\alpha^k] = \lambda_\alpha(t)U_\alpha^k, \qquad K(t)[U_\alpha^k] = \kappa_\alpha(t)U_\alpha^k, \tag{24}$$

and therefore the evolution equation is equivalent to

$$\dot{\lambda}_\alpha(t) = \int_0^t \kappa_\alpha(t-\tau)\lambda_\alpha(\tau)\,\mathrm{d}\tau. \tag{25}$$

# Memory kernels

## Theorem 8

*A memory kernel $K(t)$ generates a physical evolution if and only if*

$$\widetilde{\kappa}_\alpha(s) = -\frac{s\widetilde{\ell}_\alpha(s)}{1-\widetilde{\ell}_\alpha(s)}, \qquad \textit{where} \qquad \lambda_\alpha(t) = 1 - \int_0^t \ell_\alpha(\tau)\,\mathrm{d}\tau, \tag{26}$$

*where the functions $\ell_\alpha(t)$ satisfy the conditions*

$$\int_0^t \ell_\alpha(\tau)\,\mathrm{d}\tau \geq 0, \tag{27}$$

$$d\int_0^t \ell_\beta(\tau)\,\mathrm{d}\tau \leq \sum_{\alpha=1}^{d+1} \int_0^t \ell_\alpha(\tau)\,\mathrm{d}\tau \leq \frac{d^2}{d-1}. \tag{28}$$

The Laplace transform of $f(t)$: $\widetilde{f}(s) := \int_0^\infty f(t)e^{-st}\,\mathrm{d}t$.

*J. Williams, Laplace Transforms, George Allen & Unwin, London 1973.*

# Memory kernels

## Definition 3

Two families of completely positive maps $\{N(t), Q(t)\}$ form a **legitimate pair** if

$$N(0) = \mathbb{1}, \qquad ||\widetilde{Q}(s)||_1 < 1, \qquad \mathrm{Tr}\, Q(t)[X] + \mathrm{Tr}\, \dot{N}(t)[X] = 0. \tag{29}$$

A legitimate pair allows one to construct the dynamical map

$$\Lambda(t) = N(t) + (N * Q)(t) + (N * Q * Q)(t) + \ldots, \tag{30}$$

which is the solution if the master equation with the memory kernel

$$\widetilde{K}(s) = s\mathbb{1} - \left[1 - \widetilde{Q}(s)\right] \widetilde{N}^{-1}(s). \tag{31}$$

*D. Chruściński and A. Kossakowski, Phys. Rev. A* **94**, *020103(R) (2016).*

## Example – beyond the legitimate pairs

The generalized Pauli channel for which $Q(t)$ **is not** completely positive:

$$\Lambda(t) = \frac{1}{d+1-k} \sum_{\alpha \notin \mathcal{J}} e^{dt\mathcal{L}_\alpha}, \qquad \mathcal{J} \subset \{1, \ldots, d+1\}, \qquad |\mathcal{J}| = k. \tag{32}$$

# Memory kernels

A **semi-Markov evolution** is fully determined by a family of completely positive maps $Q(t)$ for which

$$\int_0^t Q^\dagger(\tau)[\mathbb{I}]\,\mathrm{d}\tau \leq \mathbb{I}, \qquad \mathrm{Tr}(XQ(t)[Y]) =: \mathrm{Tr}(Q^\dagger(t)[X]Y). \tag{33}$$

For the generalized Pauli channels, one finds

$$Q(t) = \frac{1}{d-1}\sum_{\alpha=1}^{d+1} f_\alpha(t)\mathbb{U}_\alpha, \qquad N(t) = \left[1 - \sum_{\alpha=1}^{d+1}\int_0^t f_\alpha(\tau)\,\mathrm{d}\tau\right]\mathbb{1}. \tag{34}$$

*D. Chruściński and A. Kossakowski, Phys. Rev. A* **95**, *042131 (2017).*

## Example

A semi-Markov evolution is provided by the convex combination of Markovian semigroups

$$\Lambda(t) = \frac{1}{d+1}\sum_{\alpha=1}^{d+1} e^{dt\mathcal{L}_\alpha}, \qquad f_\alpha(t) = \frac{d-1}{d+1}\exp\left[-\frac{d(d+1)-1}{d+1}t\right] \tag{35}$$

# Conclusions

**The most important results:**

- construction of quantum channels by imposing the condition of covariance with respect to representations of finite groups;
- necessary and sufficient conditions for admissible generators and memory kernels;
- P-divisibility conditions for the generalized Pauli channels.

**Further work:**

- mixtures of Markovian generalized Pauli dynamical maps;
- geometry of Pauli maps and generalized Pauli channels;
- generalization of Pauli channels using mutually unbiased measurements.

# Ending remarks

This PhD thesis was based on the following publications:


1. D. Chruściński and K. Siudzińska, *Generalized Pauli channels and a class of non-Markovian quantum evolution*, Phys. Rev. A **94**, 022118 (2016);
2. K. Siudzińska and D. Chruściński, *Memory kernel approach to generalized Pauli channels: Markovian, semi-Markov, and beyond*, Phys. Rev. A **96**, 022129 (2017);
3. K. Siudzińska and D. Chruściński, *Quantum channels irreducibly covariant with respect to the finite group generated by the Weyl operators*, J. Math. Phys. **59**, 033508 (2018).



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1. *Non-Markovian quantum dynamics* No. 2015/17/B/ST2/02026 (OPUS 9),
2. *Evolution of open quantum systems governed by unitarily covariant quantum channels* No. 2018/28/T/ST2/00008 (ETIUDA 6)

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