

Scientific Award

of Polish National Quantum Information Center (KCIK)

for solving selected problems in quantum information and related fields

KCIK Steering Committee
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(Dated: Sopot, December 01, 2019)

General Rules

Any scientist worldwide is eligible to participate. The **Golden KCIK Award** will be conferred for solving one of five KCIK problems on quantum information listed below. To participate, an author should send before January 31, 2020, to the email address: kcikaward@ug.edu.pl

- a single page summarizing the solution (in pdf format),
- a link to an arXiv preprint posted in 2019, in which a solution of the problem is provided.

In 2020 the Golden KCIK Award conferred for the year 2019 is 2019 EUR. Each year up to two prizes can be awarded.

Each recipient of the KCIK Award will be invited to present his/her work during the 11-th KCIK Symposium on Quantum Information, Gdańsk (Poland), May 14-16, 2020. The local costs will be covered by KCIK. The Golden Award can be conferred to a group of authors who will share the prize. In such a case the awarded group selects a single representative to present the work during the Symposium.

If a given problem will not be solved during the year $XXXX$, the competition will be reopened for the Next Year $XXXX + 1$ with the same rules. The KCIK Award will be linearly upgraded to $XXXX + 1$ EUR and the invitation to Gdańsk for the *KCIK Symposium on Quantum Information* will be valid for the year $XXXX + 2$. The Competition will be closed if all five problems are solved. Then, as always happens, new problems will come into play.

Five KCIK problems

I. DISCRETE STRUCTURES IN THE HILBERT SPACE

The space of pure quantum states of a fixed dimension¹ N is isotropic – no quantum state is "more equal" than others. However, it is legitimate to ask, for which dimensions certain particular *constellations* of quantum states with prescribed properties do exist. Although it is widely believed that symmetric informationally complete generalized quantum measurements do exist for any N , they do so in different ways – see Problem 1 and references therein.

Furthermore, if one looks for other quantum states' configurations, the answer depends on the prime decom-

position of the dimension. Two of the problems formulated below concern the dimension $N = 6$, the smallest *perfect number*, equal to the sum of its positive divisors. More importantly, six is the smallest natural number which is neither a prime nor a power of a prime. And from the quantum perspective, this is the smallest dimension, for which the problems concerning the number of mutually unbiased bases and the number of quantum orthogonal Latin squares remain open.

The list of open problems formulated for the KCIK Award and presented here consists of five items only. Why five? In view of problems 2 and 3, specified below, it looks like our understanding of the number six is still not satisfactory...

KCIK Problem 1. Construct an **infinite sequence** of SIC POVM in different dimensions, N_1, N_2, N_3, \dots

Setup. A *symmetric informationally complete positive operator valued measure* (SIC POVM) [1, 2] associated with N -dimensional Hilbert space is given by a set of N^2 vectors $|\psi_j\rangle \in \mathcal{H}_N$ satisfying relation $|\langle\psi_j|\psi_k\rangle|^2 = \frac{N\delta_{ij}+1}{N+1}$. This set defines a generalized quantum measure-

¹ While presenting the problems we used notation common in each subfield. Thus the reader is advised that the notation is not entirely consistent throughout entire document. In particular, in different problems the symbols n, d, k and N have different meaning.

ment capable to extract complete information concerning the measured density matrix, described by $N^2 - 1$ real parameters. Such a constellation of N^2 pure states forms a simplex inscribed in the set of one dimensional projectors, the convex hull of which forms the entire set of $N^2 - 1$ dimensional density matrices. For an accessible guide to the SIC problem in low dimensions consult [3].

Motivation. From a mathematical point of view, we ask about the maximal set of *complex equiangular lines* in a given dimension N . From a physical perspective one looks for an optimal scheme of a quantum measurement of an arbitrary size N . Solving the SIC existence problem for any dimension will significantly contribute to our understanding of the geometry of the set of quantum states [4].

According to the 1999 conjecture by Zauner [1], for any dimension N there exists a *fiducial vector*, such that all remaining $N^2 - 1$ elements of the desired SIC can be obtained by acting on it with unitary matrices representing elements of the Weyl–Heisenberg group. For $N \leq 28$ and several other dimensions (e.g. $N = 30, 31, 35, 37, 39, 48, 53, 124, 195, 323, 1299$) the solutions are known analytically [5–9]. Numerical solutions are known for all $N \leq 189$ and for several much larger dimensions [5–7, 10], including $N = 844, 2208$. However, in spite of the recent research effort [11–13], the general conjecture of Zauner remains unproven. Finding an infinite family of SICs, as requested in KCIK Problem 1, could become decisive step in this direction. Furthermore, let us emphasize inspiring connections to some major open questions in algebraic number theory, including the 12th problem of Hilbert [9, 11, 14, 15].

KCIK Problem 2. Construct a set of *at least 4 mutually unbiased bases (MUBs) of order six* or prove that there are no 7 MUBs in \mathcal{H}_6 .

Setup. Consider a set of K bases $\{|\psi_i^{(m)}\rangle\}$ ($1 \leq m \leq K$, $1 \leq i \leq N$) in N -dimensional complex Hilbert space \mathcal{H}_N , so that all vectors in each basis are orthogonal, (i) $\langle \psi_i^{(m)} | \psi_j^{(m)} \rangle = \delta_{ij}$. These bases are called *mutually unbiased* (MUB), if any two bases are unbiased, (ii) $|\langle \psi_i^{(m)} | \psi_j^{(n)} \rangle|^2 = 1/N$ for $m \neq n$.

It is easy to show that there exist no more than $(N + 1)$ MUBs in \mathcal{H}_N . For any $N \geq 2$, there exist at least three MUBs. If the dimension N is a prime number or a power of a prime, $N = p^k$, there exists a complete set of $(N + 1)$ MUBs [16, 17]. Several methods to construct MUBs are known [18–20] and all solutions for dimensions 2 – 5 are classified [21]. If N is a power of prime, various properties of a complete set of $(N + 1)$ MUBs are already understood [22–27], but otherwise the number of existing MUBs remains unknown [28, 29]. In particular, for $N = 6$ a complete set would consist of seven MUBs, but up till now only solutions containing three bases were found [30–38]. If a complete set of seven MUB exists, it cannot contain a triple of product bases [39, 40].

A matrix U which relates two unbiased bases of order N , belongs to the set of complex Hadamard matrices, which includes unitary matrices, such that all its entries have the same squared modulus $|U_{ij}|^2 = 1/N$. Interestingly, the set of complex Hadamard matrices is fully characterized [41, 42] for $N \leq 5$, while for higher dimensions further connections between Hadamard matrices and MUBs were found [43].

Motivation. On the one hand, finding the complete set of MUBs in dimension 6 would yield an optimal scheme of orthogonal quantum measurement in this dimension. More importantly, deciding whether such a configuration exists has significant implications for foundations of quantum theory, as up till now our understanding of basic properties of finite dimensional Hilbert spaces is not complete. On the other hand, a possible non-existence result is of a considerable mathematical interest, as it would show that the number 6 is indeed very special and ‘less equal than others’. Research on the MUB problem reveals further intricate links between foundations of quantum theory and several fields of mathematics, including Galois rings, group theory, combinatorics, finite fields and projective geometry [44–51].

KCIK Problem 3. Determine whether there exist two *quantum orthogonal Latin squares* [52, 53] of order six. In other words, find a solution of the problem of *36 entangled officers of Euler* or demonstrate that it does not exist.

Setup. A *Latin square* (LS) of size N consists of N sets of N symbols arranged in such a way that no row or column of the square contains the same number twice. The name refers to papers of Leonhard Euler [54], who used Latin characters as symbols to be arranged.

Two orthogonal Latin squares (also called Graeco-Latin squares) of size N consist of N^2 cells arranged in a square with a pair of ordered symbols in each cell, for instance one Greek character and one Latin. Every row and every column of the square contains each element of the pair exactly once, and no two cells contain the same ordered pair. It is easy to show that for a given dimension N there exist no more than $(N - 1)$ mutually orthogonal Latin squares (MOLS). This bound is saturated if N is a prime or a power of prime [55].

Euler analyzed the problem of 36 officers from six regiments, each containing 6 officers of 6 different ranks. They should be arranged before a parade into a square 6×6 such that each row and each column holds only one officer from each regiment and only one officer from each rank. Euler wrote in 1782 that this problem has no solution [54] without providing a formal proof, established only in 1901 by Gaston Tarry [56]. This result implies that there is no pair of OLS of size 6, so that the upper bound for the number of MOLS, in this case $N - 1 = 6$, is not saturated. For any $N \geq 7$ there exist at least two MOLS, in particular also for $N = 2 \times 5 = 10$ – consult a novel by Georges Perec [57]. In general, the problem

of finding the number of MOLS for any N remains open [58].

As a rule of thumb for any interesting classical notion one can find a quantum analogue. A *quantum Latin square* (QLS) is an $N \times N$ table of N^2 vectors from N dimensional Hilbert space \mathcal{H}_N arranged in such a way that every row and every column of the table forms an orthonormal basis in the space [59]. Two *quantum orthogonal Latin squares* (QOLS) are defined [52] as collection of N^2 normalized vectors from a composite space $\mathcal{H}_N \otimes \mathcal{H}_N$, which are mutually orthogonal so they form an orthonormal basis. They are arranged in an $N \times N$ table such that for every row (column) the sum of all states in each row (column) is proportional to the maximally entangled state, $|\psi_+\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^N |j\rangle \otimes |j\rangle$. Any Graeco-Latin square leads to such a design, since it suffices to treat the pair of classical objects (α, B) as a product state, $|\alpha\rangle \otimes |B\rangle$.

An n -partite pure state is called *absolutely maximally entangled* (AME) state if it is maximally entangled with respect all possible bipartitions [60], so that all its reductions consisting of k subsystems, with arbitrary $k \leq n/2$, are maximally mixed. A density matrix ρ on a given M -dimensional Hilbert space is *maximally mixed*, if it is proportional to the identity operator on this space, $\rho = \mathbb{1}_M/M$. It is known that there are no AME states of 4 qubits [61], and equivalently, a pair of QOLS does not exist for $N = 2$.

As there are no two orthogonal Latin squares (OLS) of size six, the famous classical problem of *36 officers of Euler* has no solution [62]. An analogous quantum problem, which involves 36 *entangled officers*, remains open.

Motivation. This problem can be reformulated in several other settings. Establishing a negative result is equivalent to proving that a) there is no *absolutely maximally entangled* (AME) state of four subsystems with six levels each [60, 63, 64], thus the corresponding quantum error correction code [65], written $((4, 1, 3))_6$, does not exist; b) there is no *2-unitary* matrix $U \in U(36)$ – see [66] – which saturates the absolute bound for *entangling power* [67]; c) there are no *perfect tensors* [68] with four indices, each running from 1 to 6. Furthermore, a negative result would directly imply the famous Euler conjecture that there are no two orthogonal Latin squares of size 6. On the other hand, rather unlikely, but still possible positive result, could become an important step towards development of *quantum combinatorics*: a search for particular constellations of discrete quantum objects, with special properties of symmetry and balance, hidden in the continuous Hilbert space.

As problems 2 and 3 refer to the same dimension, $N = 6$, it is natural to speculate that they might be somehow related. It seems, however, that a connection between problems of finding the maximal number of MOLSs and MUBs for a given dimension is not a direct one [49, 50].

II. QUANTUM ENTANGLEMENT AND ITS DISTILLABILITY

Any bipartite product state is called separable, while all other pure states are entangled. A density matrix (mixed state) is called *entangled* if it cannot be represented as a convex combination of product states [69].

Entanglement forms a crucial resource used in the theory of quantum information processing. Therefore, one of the major problems in this field is to decide, whether a given quantum state of a composite system is separable or entangled [70]. Up till now, such a problem is solved only for 2×2 and 2×3 systems, as in these cases the single positive partial transpose (PPT) criterion provides a constructive answer [71]. It is known [72, 73] that already for the 3×3 system any finite number of separability criteria will not allow us to settle, whether a given quantum state is entangled or not.

KCIK Problem 4. Establish whether there exist **bound entangled states with negative partial transpose (NPT)**.

Setup. A state is called *bound entangled* if it is entangled but not distillable [74–76]. A bipartite state ρ defined on a bipartite Hilbert space $\mathcal{H}_d \otimes \mathcal{H}_d$ is called *distillable*, if it is *n-copy distillable* for some finite n . The property of *n-copy distillability* means that there exist two-dimensional (i.e. of rank two) projectors P and Q such that the matrix $(P \otimes Q)[\rho^\Gamma]^{\otimes n}(P \otimes Q)$ has a negative eigenvalue [75, 77]. Here ρ^Γ stands for the *partial transpose* of the state ρ defined by its matrix elements as $\langle ij|\rho^\Gamma|kl\rangle = \langle il|\rho|kj\rangle$. One says that the state has *negative partial transpose* (NPT) iff its partial transpose ρ^Γ has some of its eigenvalues negative. It should be stressed that the projectors P and Q act on the product $(\mathcal{C}^d)^{\otimes n}$ of all n Hilbert spaces associated with left and right subsystems of copies of the considered bipartite system, respectively — see [78, 79].

The question of NPT bound entanglement is closely related to a mathematical problem concerning 2-co-positive maps [78],[80]. A linear map $\Lambda : M_d(\mathcal{C}) \rightarrow M_d(\mathcal{C})$ acting on \mathcal{H}_d is called *positive* iff it transforms any matrix with non-negative eigenvalues into a matrix with the same property. Furthermore, a linear map Λ is called *k-positive* if and only if the following extension $\mathbb{1}_k \otimes \Lambda : M_k(\mathcal{C}) \otimes M_d(\mathcal{C}) \rightarrow M_k(\mathcal{C}) \otimes M_d(\mathcal{C})$ is positive, where $\mathbb{1}_k$ stands for the identity map (i.e. the one that maps any complex matrix from $M_k(\mathcal{C})$ to itself). The map is called *completely positive* iff it is k -positive for any k . For a finite dimension d , to ensure complete positivity it is enough to check only k -positivity for $k = d$. A map Λ is called *k-co-positive* if and only if the composition $S = T \circ \Lambda$ is k -positive, where T stands for transposition.

Motivation. This is one of the long-standing open questions of quantum information theory [78, 81]. Its positive solution would have several consequences. If NPT bound entangled states exist then the set of non-distillable entangled states is neither closed under the

tensor product nor under mixing – see [80]. The latter means that there would exist two non-distillable entangled states such that their mixture were distillable. This would imply that one of the central measures of entanglement theory, namely distillable entanglement (which describes asymptotic amount of entanglement that can be distilled from many copies of a given state by local operations and classical communication [70]) is neither additive nor convex [80].

Therefore positive solution of the present problem would lead to an extremal example of *superadditivity*. Namely it has been proven that for any NPT state there exists PPT bound entangled state such that the product of the two is distillable [82]. If the NPT state were bound entangled, then we would have the pair of two bound entangled state (with distillable entanglement measure zero) such that their tensor product would be distillable (i.e. having the measure strictly positive). This is an extremal case of superadditivity: two objects containing no resource of a given type, if put together constitute a single object that, surprisingly, turns out to contain some amount of the resource (for this type of effect on the ground of quantum channel capacities see [83]).

One can show [78] that the existence of n -copy non-distillable state is equivalent to the existence of a completely positive map Λ such that it is 2-co-positive and its n -th tensor power $\Lambda^{\otimes n} = \Lambda \otimes \cdots \otimes \Lambda$ is also co-positive. Consequently, existence of NPT bound entanglement is equivalent to existence of a completely positive map that is 2-co-positive and its arbitrary high tensor power is also co-positive.

Note that for any n there exists an n -copy nondistillable state that is $(n+1)$ -copy distillable (see [84]) which may be considered as an indication that the present problem of existence of NPT bound entanglement is hard.

KCIK Problem 5. Show that the two-ququart Werner state $\rho(4, -1/2)$ is not 2-copy distillable. This state, defined below, is the only two-ququart Werner state, for which its partial transpose ρ^Γ is proportional to a unitary matrix.

Setup Consider the family of Werner states, defined on

the Hilbert space $\mathcal{C}^d \otimes \mathcal{C}^d$ as $\rho(d, \alpha) = \frac{\mathbb{1} \otimes \mathbb{1} + \alpha V}{d^2 + \alpha d}$ with the general range of the parameter $\alpha \in [-1, 1]$. The matrix V stands for the *Swap operator*, defined by its matrix elements, $\langle ij|V|kl\rangle = \delta_{il}\delta_{jk}$. The Werner states are invariant with respect to twirling with local unitaries [69], so they are also called $U \otimes U$ -invariant. It is conjectured [78, 80, 85] that Werner states which are not 1-copy distillable are also 2-copy nondistillable.

Motivation. It would be a first step for a possible proof of existence of NPT bound entanglement. Equivalently, it would provide a very elegant completely positive map that is 2-co-positive, such that a tensor product of its two copies also has that property. Interestingly, this question is equivalent to the following mathematical problem [79]: show that sum of squares of the two largest singular values is bounded by $\frac{1}{2}$ for any matrix X of the form $X = A \otimes \mathbb{1} + \mathbb{1} \otimes B$, where A and B denote traceless matrices of size 4 satisfying $\text{Tr}(A^\dagger A) + \text{Tr}(B^\dagger B) = \frac{1}{4}$. So far, the bound equal to $\frac{1}{2}$ has been proven [79] under the additional assumption that A and B are normal, so they commute with their hermitian conjugates.

Parameters $d = 4$ and $\alpha = -\frac{1}{2}$ in the problem are chosen since they correspond to the case of:

(i) the minimal dimension for which the very special Werner state is 1-copy nondistillable, namely the state $\rho(d, -\frac{2}{d})$ which has its partial transpose proportional to the dichotomic unitary operator $U = I - 2|\psi_+\rangle\langle\psi_+|$, where $|\psi_+\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^d |j\rangle \otimes |j\rangle$ denotes the maximally entangled state. A dichotomic unitary operator has eigenvalues ± 1 .

(ii) the unique dimension where the Werner state with the parameter $\alpha = -\frac{2}{d}$ is located just *on the boundary of a 1-copy nondistillability* (i.e. in case of $d = 4$ all the states with $\alpha < -\frac{1}{2}$ are already 1-copy distillable, which is not true for $d > 4$).

The choice of the state with its partial transpose proportional to the dichotomic unitary operator is motivated by the fact that checking its n -copy distillability seems to be easier than in the general case. In particular, proportionality to the dichotomic unitary operation is conserved under taking the tensor product.

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